"Gothic" characters. That the spacing into groups of threes or into groups of fives laterally and vertically has an important bearing upon legibility and quick understanding is highly probable; that the use of white spaces in the place of black rulings works great change is undoubted. Which of these, and what other features are to be preferred in tables, is of great consequence. These questions can only be answered after very exhaustive experiments of no mean order. A competent psychologist and a competent mathematician working jointly would make the best combination for attempting such a great task.

5. **Forthcoming Articles.** We are glad to announce that short articles dealing with very specific topics are already promised. Among others we may mention: an article on the teaching of solid angles; one on the desirability of accurate space drawings; one on methods of interpolation in a table of logarithms of the trigonometric functions for small angles; one on the elements of the theory of insurance. It is hoped that these and others equally definite in topic will not be long delayed. In this connection see the note at the bottom of page 140 of this issue.

E. R. HEDRICK, *for the editors.*

HISTORY OF THE EXPONENTIAL AND LOGARITHMIC CONCEPTS.

By FLORIAN CAJORI, Colorado College.

IV. FROM EULER TO WESSEL AND ARGAND. 1749–about 1800.

A HALF CENTURY OF BARREN DISCUSSION ON LOGARITHMS.

In an essay *Sur les logarithmes des quantités negatives*[1] D'Alembert refers to his correspondence of 1747 and 1748 with Euler on this subject and states that he had read Euler's paper of 1749, but felt still that the question was not settled. D'Alembert proceeds to advance arguments of metaphysical, analytical and geometrical nature which shrouded the subject into denser haze and helped to prolong the controversy to the end of the century. Defining logarithms by the aid of two progressions, as did Napier, D'Alembert asserts that the logarithms of negative numbers are not imaginary, but real, or rather, that they may be represented at will as real or imaginary, since everything hinges on the choice of the system of logarithms. As his metaphysical argument against Euler's conclusion he gives it as improbable that the logarithmic curve $y = e^x$ passes from $x = \infty$ to imaginary values. As a geometric reason he asserts that all curves for which $a^n dy/y^n = dx$, n being odd, have two branches which are symmetrical with respect to the X-axis and yielding two values, y and $-y$ for one and the same value of x. He claims his proof to be general, hence true for $n = 1$.

[1] *Opuscules mathématiques*, T. I, Paris, 1761, pp. 180–209.

This conclusion, says Karsten,[1] rests upon an error in sign, so simple that he doubts whether D'Alembert was really in earnest in his argument. The disagreement between D'Alembert and Karsten was due to a lack of a clear and consistent definition of negative areas, and to the use of infinite areas involving the indeterminate form $\infty - \infty$. No doubt D'Alembert and others had a right to assume two branches to the logarithmic curve, if they wished; the whole question is one of assumption. If we agree that $y = e^x$ shall be so interpreted that for every fractional value of x with an even denominator, both real roots shall be taken, then there are two branches, of course; if we agree to restrict ourselves to the positive root only, then there is naturally only one branch. But what most mathematicians of that time failed to understand, was Euler's remark that the question of one branch or two branches has in fact nothing to do with the question as to the nature of logarithms of negative numbers. The former is a question in evolution, the second is a question in logarithmation; the former has reference to one of the two inversions of involution, the latter to the other inversion. In his paper of 1749 Euler had explicitly confined himself to the system with the positive base $e = 2.718 \cdots$ and had concluded that the logarithms of negative numbers are complex. D'Alembert selects a negative base by taking a geometric series $1, -2, 4, -8, \cdots$ and the arithmetic series $0, 1, 2, 3, \cdots$. Thus he obtains certain negative numbers that have real logarithms. When Euler says that $l(+a) = l(-a)$ yields zero for the logarithm of all complex numbers of unit modulus, D'Alembert in reply again changes his system of logarithms to one in which the arithmetic series is $0, 0, 0, \cdots$ and reminds the reader that any geometric series whatever can be associated with this. For one and the same system of logarithms D'Alembert selects the two series

$$\cdots \quad 2n \quad n \quad 0 - n - 2n \cdots \infty \cdots - 2n - n \quad 0 \quad n \quad 2n \cdots$$
$$\cdots - b^2 - b - 1 \quad 1/b \quad 1/b^2 \cdots 0 \cdots 1/b^2 \quad 1/b \quad 1 \quad b \quad b^2 \cdots$$

in which the base is arbitrarily altered in passing from -1 to $1/b$; this is done to show that $lb^m = l(-b^m)$. What other mathematician of renown ever changed his base so often in defence of a mathematical theory!

Though not published until 1761, this paper of D'Alembert was written some years previous, certainly before the publication in 1759 of a paper, *Reflexions sur les quantitées imaginaires*[2] by Daviet de Foncenex. This paper is the more interesting because its author, a pupil of Lagrange, remarks that Lagrange had communicated to him his views on the logarithms of negative numbers. Lagrange himself never published anything on this subject. De Foncenex gives an elementary proof for $\log(\cos\varphi + i\sin\varphi) = \varphi\sqrt{-1}$, then writes $\cos\varphi + \sqrt{-1}\sin\varphi = a + b\sqrt{-1}$, and takes for φ, $\varphi + 2\lambda\pi$. Then $(\varphi + 2\lambda\pi)\sqrt{-1} = \log(a + b\sqrt{-1})$. This cotesian result is derived by de Foncenex with the aid

[1] W. J. G. Karsten, "Von den Logarithmen verneinter Grössen," 1. und 2. Abtheilung, *Abh. Münch Acad.*, V, 1768, S. 55.

[2] *Miscellanea Philosophico-mathematica societatis privatæ Taurinensis*, I, 1759, pp. 113–146.

of the hyperbola, the very curve by which J. Bernoulli I derived results contradictory to this. He says that Bernoulli did not pay proper attention to questions of continuity. In $dz/z = -du/u$, the element of surface dz/z becomes finite while u is passing from $+\infty$ to $-\infty$. He argues that, since the element of surface changes its sign without passing through zero, there is no continuous transition from positive to negative surfaces of the hyperbola; since $z = a^u$ yields $du = dz/z$, there is no continuous passage from the $+$ to the $-$ branch of the logarithmic curve; the branches are real, but isolated and algebraically independent of each other, though transcendentally connected. Apparently in contradiction of his own earlier analytical conclusions, de Foncenex now grants that the logarithms of negative numbers may be considered real, like those of positive numbers.

A reply to de Foncenex was prepared by D'Alembert, under the title, *Supplément au mêmoire prêcédent.*[1] This reply appeared at the same time and in the same volume as his first article. D'Alembert discusses questions of geometry, corrects some mistakes of de Foncenex and clings to some erroneous views of his own.

De Foncenex prepared a reply to D'Alembert's two papers,[2] in which he declares that the arithmetical side of this controversy must be kept distinct from the geometrical. If in $y = e^x$, e is a fixed number, there can be no real logarithms of negative numbers. In the study of the logarithmic curve, we must inquire into the existence of two branches and their relation to each other without using processes of integration. He finds the ordinate β of the evolute of $y = e^x$ to be $(2y^2 + 1) \div y$. Not satisfied with this, he puts $1 + y^2 = u$ and finds $\beta^2 = (4u - 4u^2 - 1) \div (1 - u)$. Hence, he says, there are two values for β or there are two branches of the evolute, both symmetrical with respect to the X-axis. Since the evolute has two branches, the curve itself must have two branches. He now admits D'Alembert's contention that the two branches are algebraically connected. He enters upon the impossible task of harmonizing results by writing $\varphi\sqrt{-1} = \log(\pm\cos\varphi \pm \sin\varphi\sqrt{-1})$, with the direction that the proper signs be chosen in every special case. It is seen that de Foncenex started out in close alignment with the views of Euler, but departed from them considerably through his study of questions of continuity of curves and his false interpretation of the roots of a quadratic equation.

D'Alembert wrote the article "Logarithme" for Diderot's *Encyclopêdie ou dictionnaire raisonné,* 9, Paris, 1765. This article was later embodied in the great *Encyclopédie méthodique,* 1785, the mathematical part of which was translated into Italian in 1800. On the subject of logarithms of negative numbers the article voiced the views of J. Bernoulli I. rather than those of Euler. The only other Frenchman who before the closing years of the eighteenth century is known to us as participating in this discussion was Louis Charles Trincano, who con-

[1] *Opuscules mathématiques,* T. I, Paris, 1761, pp. 210–230.

[2] De Foncenex, "Eclaircissemens, etc.," *Mélanges de phil. et de math. de la soc. roy. de Turin pour les années* 1760–1761, pp. 337–344.

tributed a *Mémoire sur les logarithmes des quantités négatives* as an appendix to his father's *Traité complet d'arithmétique*, Paris et Versailles, 1781. The young man says truthfully enough that if one assumes the geometric progression $-2, -4, -8, \cdots$ or $2, -4, 8, \cdots$ or $-2, 4, -8, \cdots$ then some or all negative numbers have real logarithms. According to the first progression, positive numbers would have no real logarithms. In $2, 4, 8, \cdots$ the logarithm of a negative number is *chimérique*.

Notwithstanding the influence of D'Alembert, Euler's theory acquired a firm foothold in France at the close of the century. In 1797–9 appeared in Paris Lacroix's great treatise on the calculus, which presents logarithms unreservedly from the eulerian stand-point.[1]

It is worthy of note that we have not encountered a single research, published in England in this century and touching the topic under discussion. Can this be due to the alienation between English and continental mathematicians which resulted from the Newton-Leibniz controversy on the invention of the calculus?

In Germany there appeared between 1750 and 1770 three writers who accepted the eulerian view and were unusually clear in their expositions. We refer to Charles Walmesley, an English Roman catholic prelate who was elected member of the Berlin Academy and contributed a paper to its memoirs, J. A. Segner, professor at Halle, and W. J. G. Karsten, professor at Bützow, later at Halle. Walmesley wrote an article of four pages, *Méthode de trouver les logarithmes de chaque nombre positif, négatif, ou meme impossible*,[2] which contains a brief derivation, by the integral calculus, of the formula $ac\sqrt{-1} = \pm \log (y \pm u\sqrt{-1})$, where ac is an arc of unit-circle, y and u the cosine and sine of the arc. From this formula, and the consideration of the periodicity of trigonometric functions, Walmesley deduces the Eulerian values for $\log 1$, $\log -1$, $\log \pm \sqrt{-1}$, $\log (a + b\sqrt{-1})$. In this article, as in Euler's article of 1749, y and $u\sqrt{-1}$ are tacitly taken along lines perpendicular to each other. To find the logarithms of $a + b\sqrt{-1}$, Walmesley changes it to the form $(a^2 + b^2)^{-\frac{1}{2}}(y + u\sqrt{-1})$; now he says, *il est clair qu'on doit prendre l'arc de cercle dont le sinus est u, le cosinus y*, then if the arc is φ and $\log (a^2 + b^2)^{\frac{1}{2}} = C$, we have for $\log (a + b\sqrt{-1})$ the values $C + \varphi\sqrt{-1}$, $C + (\varphi \pm 2\pi)\sqrt{-1}$, $\cdots$. We suspect that the diagram attributed to Wessel and Argand was in the *minds* of mathematicians long before it was recorded on *paper*. Walmesley closes with some general remarks disclosing an uncomfortable feeling toward the negative and imaginary. He declares it improper to speak of the ratio between a positive and a negative quantity, since ratio involves only magnitude, not quality. To mix the two ideas is like considering density, weight, etc., in comparing the volumes of two solids.

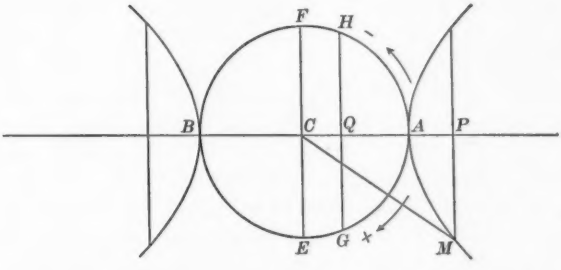
Segner lays unusual emphasis for that time upon the treatment of imaginaries and is the first to embody Euler's logarithms of complex numbers in a regular school text. In Segner's *Cursus mathematici, Pars* IV, *Halæ Magdeburgigæ*,

[1] S. F. Lacroix, *Traité du calcul dif. et calcul int.*, seconde éd., T. I, 1810, Introduction, §§81, 82.

[2] *Histoire d. l'acad. roy. d. sc. et belles-let.* [*Mémoires*], année 1755, Berlin, 1757, pp. 397–400.

1763, there is a discussion of the roots of unity, treated by De Moivre's formula, followed by a full treatment of the logarithms of complex numbers. Euler's position is here taken fearlessly. The subject is emphasized in *Pars* V (1768) in a section on *De usu imaginariorum.* Segner's treatment is analytical throughout and free from entangling alliance with the then puzzling questions relating to the logarithmic curve and hyperbola.

Karsten published an extensive article of 108 pages[1] in which D'Alembert's errors are brought out with great clearness. But Karsten agrees with D'Alembert in one point, namely that Euler's proof (quoted by us in full on p. 81) of the fundamental theorem, that $\log x$ has an infinity of distinct values, is objectionable in this respect: It conveys the idea that all logarithms belong to positive numbers, since $(1 + \omega)^n$, where ω is infinitesimal and n is infinite, appears to be always positive; the controversy itself relates to negative numbers. As a matter of fact, Euler makes no such restriction; he permits ω to be imaginary. His only error consisted in not giving an explanation sufficiently exhaustive for the needs of the time. Karsten is probably right in claiming that it would have been better, had Euler used in the proof a more elementary definition of a logarithm. Against D'Alembert, Karsten argues convincingly, that D'Alembert simply shows the possibility of systems in which negative numbers have real logarithms; the question at issue being, however, what are the logarithms of negative numbers in the natural system, with the base $e = 2.718\cdots$? Most interesting in Karsten's paper is a geometric representation of the infinitely-multiple logarithms of a number. We have never seen a reference to this ingenious construction either by contemporaries of Karsten or by historians. It looks very much as if the transactions of academies had been in some cases the safest places for the concealment of scientific articles from the scientific public.



In going over Karsten's construction it must be remembered that he wrote 29 years before Wessel. In the hyperbola $x^2 - y^2 = 1$ and the circle $x^2 + z^2 = 1$, where Karsten takes $y = z\sqrt{-1}$, the "circle is an imaginary part of the hyperbola" and *vice versa*. An arc AG of the circle may be considered as an imaginary

[1] "Abhandlung von den Logarithmen verneinter Grössen," 1. und 2. Abtheilung, *Abh. Münch. Acad.*, V, 1768.

arc of the hyperbola. Each of the four arcs which come together at A or B is a continuation of each of the others, since all are expressed by a common equation. Between each two points M and G exist numberless different arcs: MAG, $MAGBFAG$, and in general, $MAG + 2\lambda\pi$, and also $MAHBG + 2\lambda\pi$, where $\lambda = 0, 1, 2, \cdots$. To any abscissa x there belong therefore not only numberless arcs, but also numberless corresponding sectors. Let one sector of the abscissa x be $\frac{1}{2} \log (x + y) = \sqrt{-1} \cdot \frac{1}{2}$ arc cos $x = \sqrt{-1}$ sect. cos x, the sector being assumed 0 when $x = 1$. If $x > 1$, say $x = CP$, then $\frac{1}{2} \log (x + y)$ gives the sectorial areas corresponding to the arcs $AM \pm \lambda(AGBFA)$, that is, $AM \pm 2\lambda\pi\sqrt{-1}$. Only the sector ACM is real. If $x < 1$, then $x + y$ is imaginary Let $x = CQ$, then $\frac{1}{2} \log (x + y)$ measures the sectorial areas whose arcs are $AG \pm 2\lambda\pi\sqrt{-1}$, or $AFBG \pm 2\lambda\pi\sqrt{-1}$. All these arcs and sectors are imaginary. From this we see that $\frac{1}{2} \log (-1)$ is represented by the imaginary sectors whose arcs are $AEB + \lambda(BFAEB)$, or also $AFB + \lambda(BEAFB)$. These values are all included in the formula $\log (-1) = \pm (2\lambda + 1)\pi\sqrt{-1}$. "Geometry is therefore so little antagonistic to the leibnizian tenet, that the logarithms of negative numbers are impossible, as actually to give it full confirmation."[1] In this representation of logarithms there is established a double correspondence: one between points in the plane, and x and y; the other between points in the plane, and x and z, where $y = z\sqrt{-1}$.

The country in which our controversy was carried on with greatest zest during the second half of the eighteenth century is Italy. We have already referred to de Foncenex's two papers. A few years later the question was taken up by the brothers Vicenzo Riccati and Giordano Riccati. They were sons of the celebrated Giacomo Francesco Riccati, of Venice, the originator of Riccati's differential equation. Vicenzo Riccati taught mathematics at the University of Bologna. In 1767 he addressed five letters to Jacopo Pellizzari, professor in Treviso, which, we are informed, were printed along with a letter of Giordano Riccati.[2] They were reprinted twelve years later.[3] The arguments of Leibniz, J. Bernoulli, D'Alembert, Euler and de Foncenex were discussed. As a whole, Riccati considers D'Alembert's criticisms of Euler valid, when Euler takes $1 + y/n = \cos (2\lambda - 1)/n \cdot \pi \pm \sqrt{-1} \sin (2\lambda - 1)/n \cdot \pi$, where y is the required $l(-1)$, and derives from it, for $n = \infty$, $l(-1) = \pm (2\lambda - 1)\pi\sqrt{-1}$, D'Alembert modifies the process by writing $\lambda = n$, $1 + y/n = \cos (2 - 1/n)\pi \pm \sqrt{-1} \sin (2 - 1/n)\pi$. Letting $1/n = 0$ in the right member, but not in the left, he gets $1 + y/n = 1$, or $y = 0 = l(-1)$. Riccati rejects D'Alembert's revision of Euler at this point, but sustains D'Alembert in criticizing Euler's $l(1 + \omega)^n = n\omega$, n infinite and ω infinitesimal, as embracing all the logarithms, since $(1 + \omega)^n$ must be a positive number (!), hence $n\omega$ does not by his analysis represent the logarithms of negative numbers. We have seen that Euler sets no limitation as to whether the infin-

[1] Karsten, *op. cit.*, p. 103.
[2] P. Riccardi, *Biblioteca matematica Italiana*, Modena, 1893, Pt. I, Vol. II, 368.
[3] *Nuovo giornale de'letterati d'Italia*, Modena, 1779, T. XVI, pp. 137–219.

itesimal ω is real or imaginary and does not say that he means $(1+\omega)^n$ to represent a positive number. As a matter of fact, Euler puts later $(1+\omega)^n = x = -1$ and finds $l(-1)$ from this. Euler's lack of explicitness of statement at this point turned Riccati and others against him. Riccati writes $-1-\omega$ in place of $1+\omega$, thus making $l(-1-\omega) = \omega$, and arrives of course at $l(-1) = 0$. Moreover, the logarithmic curve was found to have two branches. The view of J. Bernoulli I and D'Alembert prevailed. Vicenzo Riccati wrote a letter to Joachim Pessuti, who had been in St. Petersburg and was a friend of Euler. Pessuti was then publisher of two literary journals, the *Anthologia romana* and *Effemeridi letterarie*, and after 1787 professor of applied mathematics at the Collegio della Sapienza in Rome. Pessuti defended the views of Euler in a paper, entitled *Riflessioni analitiche*, Livorno, 1777,[1] deriving Euler's main results from $u = \int \frac{dz}{1+z^2} = \frac{1}{2\sqrt{-1}} \log \frac{1+z\sqrt{-1}}{1-z\sqrt{-1}}$. Putting $z = 0$, he gets $u = n\pi$, where $n = 0, 1, 2, \cdots$. A reply to Pessuti made by V. Riccati appeared anonymously.[2] Riccati's attacks on Euler were repelled by a second writer who had come under the influence of St. Petersburg, the German astronomer Friedrich Theodor v. Schubert, in an article "Ueber die Logarithmen verneinter Grössen."[3] In the year 1778 Giuseppe Calandrelli, professor of mathematics at the Collegio Romano, attacked Pessuti and Euler in a pamphlet *Saggio analitico sopra la riduzione degli archi circolari ai logaritmi immaginarj*, Rome, 1778. As V. Riccati had done before him, so now Calandrelli casts doubts upon the correctness of the Bernoulli-Euler relation $\pi\sqrt{-1} = l(-1)$, the argument hinging on $2l(-1) = l1$. The same line of argument was pursued by P. M. Caldani, a pupil of V. Riccati and professor in Bologna. Caldani's *Della proporzione Bernoulliana fra il diametro, e la circonferenza del circolo e dei logaritmi*, Bologna, 1782, caused D'Alembert to pronounce Caldani the first mathematician in Italy.[4] Caldani objected, as Riccati had done before him, to Pessuti's writing $z\sqrt{-1} = 0$, when $z = 0$. Caldani writes $0 : 0\sqrt{-1} :: a : a\sqrt{-1}$; now, if $0 = 0\sqrt{-1}$, there follows $a = a\sqrt{-1}$, an impossibility. He wrote later a paper which we have not seen, *Riflessioni sopra un opuscolo del Padre Franceschini Barnabita, dei logaritmi de' numeri negativi*, Modena, 1791. These writers, including Pessuti, agree that Euler's proof of his fundamental theorem is not clear. When Euler writes $l(1+\omega) = \omega$, where ω is an infinitesimal, and then gets $l1 = 0$ when $\omega = 0$, he cannot at the same time claim, said the critics, that $\log 1 = 2\pi\sqrt{-1}, 4\pi\sqrt{-1}$, etc. Many of these points are discussed by the Spanish Jesuit Juan Andrès in a very interesting manner, in his book previously quoted. In 1766, when the

[1] Pessuti's paper is known to us only through the remarks on it found in Calandrelli's reply and in a book, *De studiis philosophicis et mathematicis*, Matriti, 1789, p. 215, written by JUAN ANDRÈS, and in FRÈRES (HOEFER), *Nouv. Biog. Générale*.

[2] *Nuovo giornale de'letterati d'Italia*, T. XV, Modena, 1778, pp. 144–204. See also T. XVIII, p. 107.

[3] Schubert's article is announced in the *Göttingische Anzeigen von gelehrten Sachen*, 1794, 2. Bd., p. 1313, as a reply to an article of Riccati in *Memorie della Soc. Italiana*, T. IV, p. 166.

[4] Frères, *op. cit.*, Art. *Caldani*.

Jesuits were driven out of Spain, he went to Italy and interested himself in the philosophic discussions there. He was inclined to accept Euler's conclusions, but thought that if $1 = e^0 = e^{2n\pi\sqrt{-1}}$, then we ought to have $0 = 2\pi\sqrt{-1} = 4\pi\sqrt{-1}$, etc.

Of higher type is the research of Gregorio Fontana, professor at the University of Pavia. He takes Euler's point of view. In an article, *Sopra i logaritmi delle quantità negativa e sopra gl'immaginarj*,[1] he tried to base the theory upon simpler and more rigorous proofs. He gives three proofs. First he takes $x = \text{vers } \phi$, and gets $\sqrt{-1}\, d\phi = \dfrac{dx}{\sqrt{x^2 - 2x}} = \left(\dfrac{xdx - dx}{\sqrt{x^2 - 2x}} + dx\right) \div (x - 1 + \sqrt{x^2 - 2x})$. Integrating, he gets $\phi\sqrt{-1} = \log\,(1 - x - \sqrt{x^2 - 2x})$ after letting $\phi = 0$ when $x = 0$. For $x = 2$, $\log\,(-1) = \pm\,(2n - 1)\pi\sqrt{-1}$; for $x = 0$, $\log 1 = \pm\, 2n\pi\sqrt{-1}$. A second proof starts from $x = \cos\phi$ and involves integration. A similar third proof implicates several trigonometric functions. Fontana reasons with a clearness which is unusual in articles on logarithms of that day. He tells how at one time he thought he had absolute proof of the falsity of Euler's results. In $e^{\log(-a)} = -a$ he thought that $\log(-a)$ must surely be real whenever $-a$ is real. As a matter of fact, says he, we have $\log(-a) = \pm\,(2n - 1)\pi\sqrt{-1} + \log a$. Finally Fontana derives the theorems without using the integral calculus; he uses infinite series, but pays no attention to questions of convergence.

This able article did not prevent the appearance of five "proofs" that $\log(-z) = \log z$, in Pietro Franchini's *Teoria dell'Analisi*, Rome, 1792. In 1795 Gianfrancesco Malfatti, of the University of Ferrara, discussed at length the question whether the logarithmic curve has one or two branches.[2] The equation $ydx = dy$ has an infinite number of integrals even under the restriction that $x = 0$ when $y = 1$. For any constant n, $e^{nx} = y^n$ gives rise to the differential equation $ydx = dy$. Before integrating one must agree upon the number of values y shall have. If it is to have one value, we get $e^x = y$; if two values, we get $e^{2x} = y^2$ or $(e^x - y)(e^x + y) = 0$. That $e^x = y$ has only one branch is argued from geometric and analytic considerations.

In regular mathematical text-books the lack of agreement relating to $\log(-1)$ was as marked as in the special articles. Odoardo Gherli [3]takes the ground that negative numbers differ from the positive simply in being taken in a contrary sense, and that logarithms take no cognizance of such contrarity. Pietro Paoli[4]

[1] *Memorie di matematica e fisica della società italiana*, T. I, Verona, 1782, pp. 183–202.

[2] *Memorie della reale accademia di scienze . . . di Mantova*, T. I, pp. 2–54. This article is known to us through a valuable historical monograph by Bernard F. Thibaut, *Dissertatio historiam controversiæ circa numerorum negativorum et impossibilium logarithmos sistens*, Göttingen, 1797, p. 20. Malfatti's article is reviewed in the *Göttingische Anzeigen von gelehrten Sachen*, 1796, 2. Bd., p. 1242. The *Göttingische Anzeigen* for 1786, 2. Bd., p. 1029, refers to another eighteenth century historical paper on this subject, namely the *De logarithmis numerorum negativorum* by Fridrich Mallet, professor at Upsala, which was printed in the *Nova Acta Reg. Soc. Scient. Upsaliensis*, Vol. IV, 1784, pp. 205–220.

[3] *Gli elementi teorico-pratici*, T. II, Modena, 1771, p. 297.

[4] *Elementi d'Algebra*, T. I, Pisa, 1794.

derives "il famoso Teoreme del Sig. Euler," but thinks he must use infinite series to secure rigor. Paolo Frisi, in his algebra,[3] shows familiarity with the leading writers on logarithms; he states the results reached by Euler, but is out of sympathy with them. Many results appear to him absurd, as for instance, $\log(-1) : \sqrt{-1} = \pi : 1$. He objects also to $0 \cdot \sqrt{-1} = 0$ and claims that an imaginary zero represents a real magnitude different from zero.

In a ponderous quarto volume,[2] published in 1782 at Florence, Petro Ferroni devotes a chapter to logarithms of negative numbers, only to arrive at the conclusion that analytic, as well as geometric, considerations of continuity yield $l(+1) = l(-1)$. Euler's argument, $-y = y(-1) = c^{ly} \cdot c^{\pm\lambda p\sqrt{-1}}$, hence $l(-y) = ly \pm \lambda p\sqrt{-1}$, is rejected, since by a purer doctrine (*puriore religione*) one has $-y = -c^{ly}$, and $l(-y)$ a real number. Ferroni bases his theory of general logarithms on the relation $y = \pm c^x$; he perceives no inconsistency in this procedure.

Signs of decadence are visible also in Germany. The high level reached there in the middle of the century was not maintained at its close. Kästner,[3] of the University of Göttingen, shows from the theory of ratios that we cannot have real logarithms of negative quantities. I. A. C. Michelsen, an admirer of Euler, a teacher at the royal gymnasium in Berlin, who in 1788 brought out a German translation of Euler's famous *Introductio* of 1748, added notes to that translation which indicate that Michelsen failed completely to grasp Euler's[4] argument; he holds to the view, for instance, that to every number, whether positive or negative, there corresponds one and only one logarithm. In 1795, G. S. Klügel[5] declared himself as follows: "The $\log(-x^2)$ is impossible; but the $\log(-x)$ is possible, else it must be shown that some condition does not permit x to be taken negatively." The possibility of the logarithm of a negative number rests upon the possibility of the negative number itself. Interest in the discussion of logarithms is shown in occasional notices that appeared in the *Göttingische Anzeigen von gelehrten Sachen.*[6] A notice of the year 1786 declares that the opponents of Euler tried to decide the question by formulæ of integration, radii of curvature, etc., thereby endeavoring to pass upon fundamental concepts by means of systems of notation, when, as a matter of fact, the concepts must be determined upon first, before the systems of notation are brought into action. Euler's opponents are dubbed calculators, rather than philosophers.

In conclusion, we stop to inquire, What was the result of the half-century of discussion?

---

[1] *Paulli Frisii Operum tomus primus*, Mediolani, 1782, p. 188, etc.

[2] P. Ferroni, *Magnitudinum exponentialium logarithmorum et trigonometriæ sublimis theoria nova methodo pertractata.* Florentiæ, 1782, Chap. IX.

[3] *Leipziger Magazine f. reine u. angew. Math.*, Stück IV, 1786, p. 531. See also Cantor, *op. cit.*, IV, 1908, p. 313; B. F. Thibaut, *op. cit.*, p. 21.

[4] Euler, *Einleitung in d. Analysis d. Unendlichen*, 1788, p. 503.

[5] *Hindenburg Archiv*, 1795, 3. Heft, pp. 309–319; 4. Heft, pp. 470–481.

[6] See 2. Bd., 1786, p. 1029; 2. Bd., 1792, pp. 1185, 1186; 2. Bd., 1794, pp. 1029, 1313; 2. Bd., 1796, p. 1242.

The prevailing feeling toward the theory of logarithms of negative numbers was one of hesitancy. In 1799 Christian Kramp, professor of mathematics in Strassburg, expressed doubts[1] on the correctness of the formula $l(-x) = l(-1) + lx$.

In 1801 Robert Woodhouse, of Caius College, published a paper "On the Necessary Truth of Certain Conclusions obtained by Means of Imaginary Quantities." He says:[2] "The paradoxes and contradictions mutually alleged against each other, by mathematicians engaged in the controversy concerning the application of logarithms to negative and impossible quantities, may be employed as arguments against the use of those quantities in investigation."

In 1803 L. N. M. Carnot said in his *Géométrie de position*, p. III, that the long debate has failed to clear up the paradox that, while $\log (-2)^2 = \log (2)^2$, we cannot write $2 \log (-2) = 2 \log 2$. Euler had this point cleared up in 1749; he had the assumptions, under which $2 \log (-2)$ is or is not equal to $2 \log 2$, set forth in masterly fashion, but the discussion of fifty years involved the more primitive question, whether $\log x$ is or is not equal to $\log (-x)$. Why was Euler's brilliant paper of 1749 not convincing? For three reasons:

1. Euler failed to make plain that the question, whether the logarithmic curve had one branch or two branches, had in reality nothing to do with the theory of logarithms of negative and complex numbers.

2. Euler's proof of his fundamental theorem, that $\log x$ has an infinite number of values, was not explained with sufficient fulness of detail to carry general conviction.

3. The rank and file of mathematicians had not yet learned to treat with care the many-valued inverse operations and the vexing questions of discontinuity. Nor did they avoid the danger of confusion resulting from the simultaneous consideration of logarithms of different bases.

A noteworthy feature of the long discussion deserves mention. Ordinarily one genuine proof is accepted as sufficient to establish a mathematical truth with absolute certainty; no argument can be advanced against such a proof. But, as a rule, the discussion of logarithms of complex numbers was considered as involving arguments on each side of the question, so that the decision seemed to rest on a preponderance of arguments or a preponderance of probabilities, which apparently rendered it desirable for a partisan, in lawyer's fashion, to advance as many different "proofs" as possible. This attitude was particularly noticeable in the debates carried on by correspondence. It was due mainly to a lack of sharp definition of terms and to a failure to discriminate between what is assumed without proof and what is to be subjected to rigorous demonstration.

The Union of the Logarithmic and Exponential Concepts.

The wider adoption of the definition of logarithms as exponents, in school books, was due largely to the influence and example of L. Euler, who gave it in

[1] Montferrier's *Dic. d. scienc. math.*, *T.* II, Paris, 1836, p. 185, Art. "Logarithme," where reference is made to Kramp's *Analyse des réfractions astronomiques et terrestres*, Strasbourg, 1799.

[2] *Philosoph. Transactions*, 1801, p. 111.

his *Anleitung zur Algebra.*[1] During the eighteenth century the Naperian definition, based on the two progressions, continued to be the prevailing definition. As late as 1808, C. F. Kaussler expressed decided preference for it, the new definition offering to beginners "gaps and obscurities."[2]

We have seen that Euler extended the exponential concept even to the use of imaginary exponents. We have seen that the expression $\sqrt{-1}^{\sqrt{-1}}$ is associated with his name; he recognized that it was infinitely many-valued. When the exponent is real and fractional, the multiple values of $a^{m/n}$ were recognized by Euler and others in research articles. As a rule, this many-valuedness was not discussed at all in text-books of the eighteenth century, such as the algebras of Saunderson, Blassière and Lacroix, or the mathematical texts of La Caille, J. F. Häseler, Abbé Sauri, Karsten, Reyneau, Bézout, Gherli, Rivard and Bertrand, or the mathematical dictionaries of E. Stone and Ch. Wolf.

The relation $a = e^{\log a}$ was easily deduced from the new definition of logarithms and was used by some writers of this century, such as G. Fontana in 1782.

[Further instalments will follow, carrying this history through the nineteenth century. EDITORS.]

SOME INVERSE PROBLEMS IN THE CALCULUS OF VARIATIONS.

By E. J. MILES, Yale University.

In the so-called inverse problem[1] of the calculus of variations a doubly infinite family of curves

$$y = f(x, \alpha, \beta) \tag{1}$$

is given and it is required to find a function $F(x, y, y')$ such that the given system of curves forms the system of extremals for the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')dx. \tag{2}$$

Darboux[2] has shown that the problem always has an infinite number of solutions which can be reduced to quadratures. For if in the Euler differential equation

$$F_y - F_{y'x} - y'F_{y'y} - y''F_{y'y'} = 0$$

the expression $G(x, y, y')$, where $y'' = G(x, y, y')$ is the differential equation

[1] L. Euler, *Anleitung zur Algebra*, St. Petersburg, 1770, 1. Theil, Cap. 21.

[2] C. F. Kaussler, *Lehre von den Logarithmen*, Tübingen, 1808, preface. Quoted by Tropfke Gesch. d. Elem.-Math., 2. Band, Leipzig, 1903, p. 142.

[1] For a treatment of this problem see for instance Bolza: *Vorlesungen über Variationsrechnung*, § 6c.

[2] Darboux: *Theorie des Surfaces*, vol. III, Nos. 604, 605.

from which the curves (1) are obtained, is substituted for y'' then the resulting equation

$$F_y - F_{y'x} - y'F_{y'y} - G(x, y, y')F_{y'y'} = 0 \tag{3}$$

is satisfied identically in x, y and y'. Differentiating (3) as to y' and placing $M = F_{y'y'}$ a linear partial differential equation of first order is obtained, viz:

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + G\frac{\partial M}{\partial y'} + G_{y'}M = 0. \tag{4}$$

From this M may be obtained and then F by two quadratures.

Few examples illustrating this theory have been given. Hamel[1] has studied the case when the given curves are straight lines and Stromquist[2] the case when they are circles with center on the x-axis. The present note is intended to add further illustrative examples by considering first the case when the given curves are catenaries, secondly the case when the given curves are parabolas whose axes coincide with the x-axis, and finally the case when they are the ellipses or hyperbolas $x^2/\alpha + y^2/\beta = 1$.

§1. When the Given Curves are Catenaries.

It will be assumed that the catenaries have the x-axis for directrix. They therefore have the equation

$$y = \alpha \cosh \frac{x - \beta}{\alpha}.$$

The differential equation[3] from which this equation is obtained is

$$y'' = \frac{1 + y'^2}{y}.$$

Substituting this value of y'' the Euler equation then gives the following identity in x, y and y'

$$F_y - F_{y'x} - y'F_{y'y} - \frac{1 + y'^2}{y}F_{y'y'} = 0. \tag{5}$$

Hence M in this case must be obtained from the partial differential equation

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + \frac{1 + y'^2}{y}\frac{\partial M}{\partial y'} + \frac{2y'}{y}M = 0. \tag{6}$$

---

[1] Hamel: *Über die Geometrien in denen die Geraden die Kurzesten sind;* dissertation, Göttingen, 1901; *Mathematische Annalen*, vol. 57 (1903), pp. 231–264.

[2] Stromquist: "*On geometries in which circles are the shortest lines,*" *Transactions of the American Mathematical Society*, vol. 7 (1906), pp. 175–183.

[3] This is readily verified when it is remembered that the projection of the ordinate of any point of the catenary on the normal to the catenary at this point is constant. Stated analytically this says that $y/\sqrt{1 + y'^2} = \alpha$, whence the above result.

To integrate this place $M = e^s$ and equation (6) becomes

$$\frac{\partial z}{\partial x} + y' \frac{\partial z}{\partial y} + \frac{1+y'^2}{y} \frac{\partial z}{\partial y'} + \frac{2y'}{y} = 0. \tag{7}$$

Then if Φ_1 denotes an arbitrary function of the arguments indicated, the most general solution of equation (7) is found to be

$$\Phi_1 \left\{ \frac{y}{\sqrt{1+y'^2}}, \quad x - \frac{y}{\sqrt{1+y'^2}} \log \left(\sqrt{1+y'^2} \pm y'\right), \quad z + \log\left(1+y'^2\right) \right\} = 0$$

or

$$z + \log\left(1+y'^2\right) = \Phi_2 \left\{ \frac{y}{\sqrt{1+y'^2}}, \quad x - \frac{y}{\sqrt{1+y'^2}} \log \left(\sqrt{1+y'^2} \pm y'\right) \right\}.$$

Hence M assumes the form

$$M = \frac{1}{1+y'^2} \Phi \left\{ \frac{y}{\sqrt{1+y'^2}}, \quad x - \frac{y}{\sqrt{1+y'^2}} \log \left(\sqrt{1+y'^2} \pm y'\right) \right\},$$

where again the function Φ is an arbitrary function of its arguments.

From this it follows that a first integration gives

$$\frac{\partial F}{\partial y'} = \int \frac{1}{1+y'^2} \Phi \left\{ \frac{y}{\sqrt{1+y'^2}}, \quad x - \frac{y}{\sqrt{1+y'^2}} \log \left(\sqrt{1+y'^2} \pm y'\right) \right\} dy' + \lambda(x, y)$$

$$= \int_0^{y'} \frac{1}{1+t^2} \Phi \left\{ \frac{y}{\sqrt{1+t^2}}, \quad x - \frac{y}{\sqrt{1+t^2}} \log \left(\sqrt{1+t^2} \pm t\right) \right\} dt + \lambda(x, y).$$

Finally on integrating a second time F is found to be of the form

$$F = \int_0^{y'} (y' - t) \frac{1}{1+t^2} \Phi \left\{ \frac{y}{\sqrt{1+t^2}}, \quad x - \frac{y}{\sqrt{1+t^2}} \log \left(\sqrt{1+t^2} \pm t\right) \right\} dt + \lambda(x, y)y' + \mu(x, y), \tag{8}$$

in which λ and μ are arbitrary functions of x and y.

Now every F which satisfies equation (5) necessarily satisfies equation (6) but not conversely. It is therefore necessary to find what conditions the functions λ and μ must satisfy in order that the function F given by (8) may satisfy the equation (5). When this value of F is substituted in equation (5) it must result in a function independent of y'. Consequently it may be evaluated by giving y' any definite value, say $y' = 0$. This gives us the relation

$$-\frac{1}{y} \Phi(y, x) + \mu_y - \lambda_x = 0. \tag{9}$$

If then $\bar{\lambda}$ and $\bar{\mu}$ are particular values of λ and μ which satisfy condition (9), the most general values of λ and μ are

$$\lambda = \bar{\lambda} + \frac{\partial V}{\partial y}, \quad \mu = \bar{\mu} + \frac{\partial V}{\partial x},$$

where V is an arbitrary function of x and y. If one of the functions $\bar{\lambda}$ and $\bar{\mu}$, say $\bar{\mu}$, is chosen as zero, equation (9) gives

$$\frac{\partial \bar{\lambda}}{\partial x} = -\frac{1}{y}\Phi(y, x),$$

and therefore

$$\bar{\lambda} = -\frac{1}{y}\int_{x_0}^{x_1} \Phi(y, x)dx.$$

Hence the function F has as its final form the following:

$$F = \int_0^{y'} (y' - t)\frac{1}{1+t^2}\Phi\left\{\frac{y}{\sqrt{1+t^2}}, \quad x - \frac{y}{\sqrt{1+t^2}}\log\left(\sqrt{1+t^2} \pm t\right)\right\} dt$$
$$- \frac{y'}{y}\int_{x_0}^{x} \Phi(y, x)dx + y'\frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

This is the most general form which it can assume and have as extremals the given set of catenaries.

Suppose now that a further restriction is made, namely, that the extremals are to be perpendicular to their transversals. In order to assertain what the restriction is in this case it is only necessary to recall the general condition of orthogonality of extremals and transversals, viz:

$$F(x, y, y') = \psi(x, y)\sqrt{1 + y'^2}.$$

In case of the present problem this leads to the condition

$$\Phi\left\{\frac{y}{\sqrt{1+y'^2}}, \quad x - \frac{y}{\sqrt{1+y'^2}}\log\left(\sqrt{1+y'^2} \pm y'\right)\right\} = \frac{\psi(x, y)}{(1+y'^2)^{\frac{1}{2}}}.$$

But in order that there may be some relation between the expression $\psi(x, y)/\sqrt{1 + y'^2}$ and the arguments of Φ it is necessary that their Jacobian vanish identically in x, y and y', i. e.,

$$\begin{vmatrix} \frac{\psi_x}{\sqrt{1+y'^2}} & \frac{\psi_y}{\sqrt{1+y'^2}} & -\frac{y'\psi}{(\sqrt{1+y'^2})^3} \\ 0 & \frac{1}{\sqrt{1+y'^2}} & -\frac{yy'}{(\sqrt{1+y'^2})^3} \\ 1 & -\frac{1}{\sqrt{1+y'^2}}\log\left(\sqrt{1+y'^2} \pm y'\right) & \frac{yy'}{(\sqrt{1+y'^2})^3}\log\left(\sqrt{1+y'^2} \pm y'\right) \mp \frac{y}{1+y'^2} \end{vmatrix} \equiv 0.$$

Expanding this determinant and grouping like powers of y' it is found that the following equation results:

$$\pm\, y\psi_x + y'[y\psi_y - \psi] \equiv 0.$$

Hence

$$y\psi_x = 0, \qquad y\psi_y - \psi = 0$$

and therefore

$$\psi = ky.$$

Hence if the extremals are to be catenaries which are orthogonal to their transversals the function Φ must assume the form

$$\frac{ky}{\sqrt{1 + y'^2}}.$$

§ 2. The Case of Parabolas.[1]

Let the equation of the parabolas be taken in the form

$$y^2 = 2\alpha x + \beta.$$

Then

$$y'' = -\frac{y'^2}{y}$$

and equation (3) becomes in this case

$$F_y - F_{y'x} - y'F_{y'y} + \frac{y'^2}{y} F_{y'y'} = 0. \tag{10}$$

Hence M is determined by the following equation:

$$\frac{\partial M}{\partial x} + y' \frac{\partial M}{\partial y} - \frac{y'^2}{y} \frac{\partial M}{\partial y'} - \frac{2y'}{y} M = 0. \tag{11}$$

Integrating by the method of the preceding section it is easily verified that the most general solution of equation (11) is

$$M = y^2\Phi(yy',\ y^2 - 2xyy'),$$

where again Φ is an arbitrary function of its arguments.

Hence

$$\frac{\partial F}{\partial y'} = \int_0^{y'} y^2\Phi(yt,\ y^2 - 2xyt)dt + \lambda(x, y)$$

and therefore

[1] Only points in the upper half plane are considered, for the tangent at the vertex of the parabola $y^2 = 2\alpha x + \beta$ is infinite and this case cannot be treated by the usual methods of the calculus of variations when the curve is taken in the form $y = y(x)$. If then $x_0 \neq x_1$, and $y_0 \neq y_1$, there passes one and but one parabola of the above type through any two points in the upper half plane.

$$F = \int_0^{y'} (y' - t)y^2\Phi(yt, y^2 - 2xyt)dt + y'\lambda(x, y) + \mu(x, y),$$

where λ and μ are functions of x and y.

When this value of F is substituted in equation (10) giving an equation independent of y' it is found upon evaluating by assigning the value zero to y' that λ and μ must satisfy the relation

$$\frac{\partial\lambda}{\partial x} = \frac{\partial\mu}{\partial y}.$$

Hence the most general expressions for λ and μ are

$$\lambda = \frac{\partial V}{\partial y}, \quad \mu = \frac{\partial V}{\partial x},$$

V being an arbitrary function.

The most general solution of the problem is therefore

$$F = \int_0^{y'} (y' - t)y^2\Phi(yt', y^2 - 2xyt)dt + y'\frac{\partial V(x, y)}{\partial y} + \frac{\partial V(x, y)}{\partial x}.$$

§ 3. When the Given Curves are Ellipses or Hyperbolas.

In this case the equations of the curves will be taken in the form[1]

$$\frac{x^2}{\alpha} + \frac{y^2}{\beta} = 1.$$

The differential equation belonging to these curves is found to be

$$y'' = \frac{yy' - xy'^2}{xy}.$$

Hence the partial differential equation from which M is obtained is

$$\frac{\partial M}{\partial x} + y'\frac{\partial M}{\partial y} + \frac{yy' - xy'^2}{xy}\frac{\partial M}{\partial y'} + \frac{y - 2xy'}{xy}M = 0. \tag{12}$$

By the process previously described the following expression is found for M

$$M = \frac{y}{x(y - xy')}\Phi\left(\frac{-y'}{x(y - xy')}, \frac{1}{y(y - xy')}\right).$$

[1] Here again only points in the upper half plane are considered. Then if the point (x_1, y_1) is not on either of the lines

$$x + \frac{x_0}{y_0}y = 0, \quad x - \frac{x_0}{y_0}y = 0,$$

it follows that there is a unique determination of the constants $1/\alpha$ and $1/\beta$. Hence, barring these exceptional cases, it follows that through any two points of the upper half plane there passes one ellipse or one hyperbola defined by the equation $x^2/\alpha + y^2/\beta = 1$.

Hence by quadrature

$$F = \int_0^{y'} (y' - t) \frac{y}{x(y - xt)} \Phi\left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)}\right) dt + y'\lambda(x, y) + \mu(x, y).$$

It is easily proved that the functions λ and μ must again satisfy the condition

$$\frac{\partial \lambda}{\partial x} = \frac{\partial \mu}{\partial y}.$$

Therefore the most general solution for F is seen to be

$$F(x, y, y') = \int_0^{y'} (y' - t) \frac{y}{x(y - xt)} \Phi\left(\frac{-t}{x(y - xt)}, \frac{1}{y(y - xt)}\right) dt + y' \frac{\partial V}{\partial y} + \frac{\partial V}{\partial x}.$$

Furthermore it can be proved that in neither of the last two cases can the function Φ be so chosen that the transversals are perpendicular to the extremals.

THE PROBABILITY INTEGRAL DEDUCED BY MEANS OF DEVELOPMENTS IN FINITE FORM.

By EDWARD L. DODD, University of Texas.

In modern mathematics, expansions in *finite* form are constantly coming into greater favor in preference to infinite series from which the terms after the first two or three have been dropped.

The object of this paper is to express in modern form the deduction of the probability integral as given by J. Bertrand,[1] whose method follows Laplace's,[2]—though his integral is not quite the same.

Let p be the probability that an event will happen in a single trial, where $0 < p < 1$; and let $q = 1 - p$. Then the probability, P_r, that the event will happen exactly r times on s trials is

$$P_r = \frac{s!}{r!\,(s - r)!} p^r q^{s-r}. \tag{1}$$

Here r is a positive integer not greater than s, or r is zero, accepting unity as the value of 0!

Now let

$$l = sp - r. \tag{2}$$

Then $r = sp - l$, and

$$P_r = \frac{s!}{(sp - l)!(sq + l)!} p^{sp-l} q^{sq+l}. \tag{3}$$

[1] *Calcul des Probabilités* (1889), p. 76.

[2] *Œuvres Complètes*, VII, Theorie Analytique des Probabilités, p. 284.

Stirling's[1] Formula, in *finite* form is

$$n! = n^n e^{-n} \sqrt{2\pi n}\, e^{\theta/12n}, \quad 0 < \theta < 1. \tag{4}$$

Using (4), and taking r as a positive integer *less* than s, (3) becomes

$$P_r = \frac{1}{\sqrt{2\pi spq}} \frac{1}{\left(1 - \frac{l}{sp}\right)^{sp-l+\frac{1}{2}}} \frac{1}{\left(1 + \frac{l}{sq}\right)^{sq+l+\frac{1}{2}}} e^{\frac{\theta}{12s} - \frac{\theta'}{12(sp-l)} - \frac{\theta''}{12(sq+l)}}. \tag{5}$$

Now let any positive number c be chosen; and then let s be taken large enough so that

$$sp - c\sqrt{s} > 0, \quad \text{and} \quad sp + c\sqrt{s} < s. \tag{6}$$

And let r be such that

$$sp - c\sqrt{s} \leqq r \leqq sp + c\sqrt{s}. \tag{7}$$

Then

$$|\, l \,| \leqq c\sqrt{s}. \tag{8}$$

For brevity, set

$$u = \left(1 - \frac{l}{sp}\right)^{sp-l+\frac{1}{2}}, \quad v = \left(1 + \frac{l}{sq}\right)^{sq+l+\frac{1}{2}}.$$

Now, if $|\, t \,| < \frac{1}{2}$,

$$\log_e (1 + t) = t - \frac{t^2}{2} + \theta_1 t^3, \quad 0 < \theta_1 < 1. \tag{9}$$

But, if s is sufficiently great, l/sp and l/sq will be numerically less than $\frac{1}{2}$, because of (8). Then, by the use of (9), it may be shown that

$$\log_e uv = \frac{l^2}{2spq} + R, \qquad |\, R \,| < \frac{k}{\sqrt{s}},$$

in which k is some constant; that is, k is independent of s. Indeed, to get an upper bound for R, let all its terms be made positive, let the θ_1 in (9) be taken as unity, and let l be replaced by $c\sqrt{s}$, as in (8). In the expression thus obtained, the numerators are functions of c, and each denominator contains the factor $\sqrt{s}$.

Then,

$$uv = e^{\frac{l^2}{2spq}} \cdot e^{\frac{\theta_2 k}{\sqrt{s}}}, \quad -1 < \theta_2 < 1.$$

Now if $|\, t \,| < \frac{1}{2}$,

$$e^t = 1 + 2\eta t, \quad 0 < \eta < 1. \tag{10}$$

Hence, when s is large enough,

$$e^{-\frac{\theta_2 k}{\sqrt{s}}} = 1 + \frac{2\eta' k}{\sqrt{s}}, \quad -1 < \eta' < 1.$$

[1] See Cesaro, *Corso di analisi algebrica*, Turin, 1884, pages 270 and 480; or Broggi, *Traité des Assurances sur la Vie*, Trad. S. Lattès, 1907, p. 54.

The relation (10) can also be applied to the last factor in (5), noticing that in $sp - l$, the l becomes insignificant in comparison with sp when s is large, because of (8). Hence

$$P_r = \frac{1}{\sqrt{2\pi spq}} e^{-\frac{r^2}{2spq}} + \epsilon_1, \quad |\epsilon_1| < \frac{K_1}{s}, \tag{11}$$

in which K_1 is some constant; that is, K_1 is independent of s.

For brevity, set

$$h = \frac{1}{\sqrt{2spq}}. \tag{12}$$

Let r_1 and r_2 be any two values of r which satisfy (7), with $r_1 > r_2$; and let l_1 and l_2 correspond by (2). Then,

$$\sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}} \sum_{l_1}^{l_2} e^{-h^2 l^2} + \epsilon_2, \quad |\epsilon_2| < \frac{K_2}{\sqrt{s}}, \tag{13}$$

in which K_2 is some constant.

By dropping ϵ_1 in (11), we get an approximation, P_r', for P_r; viz.,

$$P_r' = \frac{h}{\sqrt{\pi}} e^{-h^2 l^2}. \tag{14}$$

Since r is an integer, any two consecutive values of l differ from each other by unity, because of (2). Use the values of l as abscissas and the corresponding values of P_r' as ordinates. With each of these ordinates as a central line, construct a rectangle with its base upon the x-axis, the base being unity in length. The *probability curve*, whose equation is

$$y = f(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}, \tag{15}$$

passes through the middle points of the upper bases of these rectangles. The probability curve has but two points of inflection, and near the origin is concave towards the x-axis. For this portion of the curve, the following figure may be drawn. See page 126.

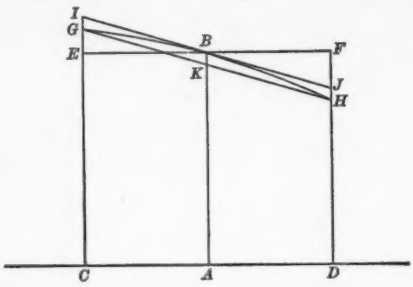
In this figure, A is the point on the X axis, whose abscissa is l; AB the ordinate whose length is P_r'; $CDFE$ the rectangle—bisected by AB—with its area numerically equal to P_r'; GBH an arc of the probability curve (15); GH a chord; IBJ the tangent to GBH at B; and K the intersection of AB and GH.

It is now to be proved that if l satisfies (8),

$$P_r' = \frac{h}{\sqrt{\pi}} \int_{l-1/2}^{l+1/2} e^{-h^2 x^2} dx + \epsilon_3, \quad |\epsilon_3| < \frac{K_3}{s\sqrt{s}}, \tag{16}$$

in which K_3 is some constant.

Proof.—In the figure, the trapezoid $CDJI$ has an area numerically equal to P_r'; and this differs from the area beneath the arc GBH by less than the area of the trapezoid $GHJI$. This last mentioned area is $\overline{KB} \times \overline{CD} = \overline{KB}$; since $\overline{CD}$ is of unit length. Hence $\overline{KB}$ is numerically greater than ϵ_3 in (16).



But

$$\overline{KB} = f(l) - \tfrac{1}{2}[f(l - \tfrac{1}{2}) + f(l + \tfrac{1}{2})]$$

$$= \frac{he^{-h^2l^2}}{2\sqrt{\pi}}\left[2 - e^{-\frac{h^2}{4}}(e^{h^2l} + e^{-h^2l})\right]. \quad (17)$$

Moreover, if $|t| < \frac{1}{2}$,

$$e^t = 1 + t + \zeta t^2, \qquad 0 < \zeta < 1. \quad (18)$$

And, from (8) and (12), it is evident that $|h^2l|$ becomes less than $\frac{1}{2}$ when s is large enough. Using (18) upon the parenthesis in (17) and also upon $e^{-h^2/4}$, and noticing that $e^{-h^2l^2} \leqq 1$, we prove (16). Beyond the points of inflection the curve (15) is convex towards the X axis; but here, also, a similar treatment applies.

Then, from (11), (14), and (16), we can conclude that

$$\sum_{r_2}^{r_1} P_r = \frac{h}{\sqrt{\pi}} \int_{l_1 - \frac{1}{2}}^{l_2 + \frac{1}{2}} e^{-h^2x^2} dx + \epsilon_4, \quad |\epsilon_4| < \frac{K_4}{\sqrt{s}}, \quad (19)$$

in which K_4 is some constant. The discussion has been carried out under the restriction imposed by (8); viz.,

$$-c\sqrt{s} \leqq l_1 < l_2 \leq c\sqrt{s};$$

but the effect of this restriction may be removed by choosing c large enough.

Indeed, let $\alpha = l_2 + \frac{1}{2}$, suppose this positive and nearly equal to $c\sqrt{s}$; and, for convenience, suppose that we can take $l_1 - \frac{1}{2} = -\alpha$. Then the integral in (19) becomes $\Theta(h\alpha)$. Now, by (12), $\alpha < c\sqrt{s}$, provided

$$h\alpha < \frac{c}{\sqrt{2pq}}.$$

But, by taking c large enough, we may take as large values of $h\alpha$ as we wish, making $\Theta(h\alpha)$ differ from unity by as little as we please. For example, if we take $c = 5\sqrt{2pq}$, we may take s and α so that $4.9 < h\alpha < 5$; and $\Theta(h\alpha)$ will differ then from unity by less than 10^{-11}. Hence, by properly choosing c and then s, and taking r_2 and r_1 nearly equal to $sp - c\sqrt{s}$ and $sp + c\sqrt{s}$, respectively, $\sum_{r_2}^{r_1} P_r$ in (19) may be made as near unity as we please. But, by (1), $\sum_0^s P_r = (p + q)^s = 1$. Hence, the rectangles which the restriction (8) excludes are negligible when s is large enough.

By reference to (11), it is evident that no essential change in (19) is made if $l_1 - \frac{1}{2}$ and $l_2 + \frac{1}{2}$ are replaced by l_1 and l_2. respectively

THEOREM: *Let p be the probability that an event will occur in a single trial, where $0 < p < 1$; let s be the number of trials to be made; and let $h = [2sp(1-p)]^{-\frac{1}{2}}$. Then the probability that the number of occurrences of the event will be some number in the set, $sp - l_2$ to $sp - l_1$ inclusive, where $0 \leqq sp - l_2 \leqq sp - l_1 \leqq s$, is*

$$\frac{h}{\sqrt{\pi}} \int_{l_1}^{l_2} e^{-h^2x^2} dx + \delta,$$

where $\lim_{s=\infty} \delta = 0$ *uniformly.*

While this commonly accepted theorem is the most important result of the foregoing set of inequalities, the relations, (11), (13), (16), and (19), are not without some individual importance.

WESTERN MEETINGS OF MATHEMATICIANS

The American Mathematical Society has been for nearly a quarter of a century the official organization through which scientific activity in mathematics has found expression in this country. In 1894 The New York Mathematical Society, which had then been in existence for six years, was expanded into the American Mathematical Society, and the *Bulletin*, which had been published for three years by the former, was continued by the latter as "A Historical and Critical Review of Mathematical Science."

The regular meetings of the Society were held in New York, except in the case of the summer meetings, which, from the first in 1894 to the nineteenth in 1912, were held respectively in Brooklyn, N. Y., Springfield, Mass., Buffalo, N. Y., Toronto, Can., Boston, Mass., Columbus, O., New York, N. Y., Ithaca, N. Y., Evanston, Ill., Boston, Mass., St. Louis, Mo., Williamstown, Mass., New Haven, Conn., Ithaca, N. Y., Urbana, Ill., Princeton, N. J., New York, N. Y., Poughkeepsie, N. Y., and Philadelphia, Pa. It thus appears that of the nineteen summer meetings, four have been held in the middle west, namely, at Columbus, in 1899, at Evanston in 1902, at St. Louis in 1904, and at Urbana in 1908.

Another notable gathering of mathematicians took place at Chicago in 1893, namely, the International Mathematical Congress held in connection with the World's Columbian Exposition. The papers read at this meeting, including nine by Americans and thirty by representatives of other countries, were published for the Society by the Macmillan Company in 1896. There was also held at Evanston, Ill., from August 28 to September 9, 1893, a Colloquium in connection with the World's Fair, consisting of a series of lectures delivered by Professor Felix Klein, of Göttingen, Germany. These lectures have been published by the Society. Other colloquia have been held from time to time in connection with the summer meetings, at which series of lectures have been delivered by members of the Society, but none of these have as yet been held in the west. The next one, however, is to be at Madison, Wis., in September, 1913, in connection with the twentieth summer meeting of the Society.

In 1896 a conference was called in Chicago by an informal committee of twenty-eight members of the Society to consider the desirability of organizing a Section of the Society in the middle west, in order that the stimulus and inspiration arising from attendance upon frequent meetings might be brought within the reach of those in this vicinity, it being quite impossible for most members in the west to attend the meetings in New York. This conference was held at the University of Chicago, in December, 1896, when a temporary organization was effected and fourteen papers were read. A second conference was called April 24, 1897, at the University of Chicago, when a communication from the Council of the Society was received authorizing the formation of a Section. The formal organization was thus completed and Professor E. H. Moore was made chairman, and Professor T. F. Holgate secretary, of the Chicago Section of the American Mathematical Society. Ten papers were read at this meeting Since that time thirty-one meetings of the Chicago Section have been held twenty-four at the University of Chicago, five at Northwestern University, one at Armour Institute, and one at the University of Minnesota in affiliation with the American Association for the Advancement of Science. The total attendance upon these meetings has aggregated 930 members and 288 visitors, and the total number of papers read is 569.

Two other Sections of the Society have been formed in the west, namely, the San Francisco Section, which meets alternately at the University of California and at Stanford University, and the Southwestern Section, which meets at various points including St. Louis, Mo., Columbia, Mo., Lawrence, Kans., and Lincoln, Neb. The latter was organized in November, 1907, and has held in all six meetings at which 79 papers have been read. The former was organized in May, 1902, and has held twenty-two meetings at which 182 papers have been read. The scientific activity shown by the records of these sectional meetings has amply justified the formal action of the Society as stated in By-Law IV, which says:

> "Whenever it shall appear to the Council that a sufficient number of members of the Society are desirous of conducting in any locality periodic meetings for the reading and discussion of mathematical papers, the Council may authorize the formation of a Section to be composed at each sectional meeting of such members of the Society as may be present; and the Council shall have the right to withdraw such authorization."

On two notable occasions regular meetings of the Society as a whole have been held in the middle west, and the meetings of the Chicago Section, which would have occurred at the same times and places, were merged in those of the Society. One of these meetings was held in Chicago, on the invitation of the Chicago Section, in April, 1911. It was the first regular meeting of the Society, aside from the summer meetings, to be held away from New York City since its founding there in 1888. It was arranged that this reunion of the eastern and western members should be especially marked by the delivery of President Bôcher's retiring address, which had been postponed from the annual meeting in the preceding December. The attendance upon this meeting numbered 115

persons, including 88 members of the Society who had come from as far east as Massachusetts, as far west as Wyoming, as far north as Minnesota and as far south as Missouri. There were 53 papers read at this meeting aside from Professor Bôcher's address on "Charles Sturm's published and unpublished work on differential and algebraic equations."

The other occasion when a regular meeting of the Society was held in the west was the Cleveland meeting in December, 1912, in affiliation with the American Association for the Advancement of Science.[1] There were 54 papers read in the mathematical sessions on this occasion and 62 members were in attendance.

Perhaps the most important occasion in connection with these gatherings was the series of meetings of mathematicians and engineers held at the University of Chicago in December, 1907, under the auspices of the Chicago Section of the American Mathematical Society, and conjointly with Sections A and D (mathematics and engineering) of the American Association for the Advancement of Science. The sole topic of these meetings was the discussion of the teaching of mathematics to students of engineering, by mathematicians, professors of engineering, and those engaged in the practice of engineering. The attendance was large and representative, including 100 mathematicians and fifty engineers from 23 state universities and technical schools and 12 other universities and colleges. The presentation of papers occupied three half days and resulted in the appointment of a committee of twenty members with Professor E. V. Huntington, of Harvard University, as chairman, to take into consideration the whole question of the mathematical curriculum in technical schools and in technical departments of colleges and universities, and to make their report to the Society for the Promotion of Engineering Education. That report has since been completed in the form of a syllabus of mathematical courses for students of engineering, which can be purchased through the aforesaid Society at 43 E. Nineteenth St., New York City.

The last meeting of the Chicago Section was held at the University of Chicago, March 21–22, 1913. There were 39 papers read during the four half-day sessions and the total attendance numbered 81 persons of whom 51 were members of the Society. The former officers of the Section were reëlected, namely, Professor D. R. Curtiss, Chairman, Professor H. E. Slaught, Secretary, and Professor A. L. Underhill, third member of the Program Committee. The sentiment of the Section was formally expressed in a resolution favoring the custom of having a special address by the chairman upon his retirement from office. The next occasion of the kind will occur at the December meeting, 1913. The last address of this kind was two years ago when Professor L. E. Dickson spoke on the "History of the representations of numbers as the sum of squares." It was the feeling of the members that we need more papers giving a general survey of some broad field and that this custom will serve, to some extent, to meet this desire.

A characteristic feature of the Chicago Section meetings is the full and free discussion of all papers presented, and especially the regular topic on the pro-

[1] Reported by Professor G. A. Miller in the February ,1913, issue of the MONTHLY.

gram called "Informal Notes and Queries," under which heading opportunity is given for discussion of any questions of mathematical interest that may not arise in connection with the formal papers. Another feature which is emphasized in these meetings is the dinner on the evening between the two days assigned to the program. These dinners are held at the Quadrangle Club of the University of Chicago where ample facilities are provided for social intercourse throughout the evening.

In closing, it should be said that, while the various sections were organized to accommodate members in their attendance upon meetings, there are no designated members of any given sections, but the membership at any meeting of a section consists of those members of the Society who happen to be in attendance. Thus a nation-wide brotherhood of mathematicians is included in the one American Mathematical Society.

H. E. SLAUGHT.

BOOK REVIEWS.

W. H. BUSSEY, CHAIRMAN OF THE COMMITTEE.

Solid Geometry developed by the Syllabus Method. By EUGENE RANDOLPH SMITH. American Book Company, New York, 1913. xii + 211 pages. $0.75.

This is a book for the teacher of solid geometry who wishes to try something out of the ordinary. The propositions are not proved in full and formal fashion, but in the opinion of the author the book contains as much suggestion and guidance as the student needs. The aim of the book is to encourage original thinking and to reduce mere memory work to a minimum. There are three preliminary chapters, one on the relation between solid geometry and plane geometry, one on methods of attack, and one on the representation of solid figures by drawing. At the end of the book there are 23 pages of college examination questions selected from the questions set by the College Entrance Examination Board and by various colleges.

W. H. B.

Memoranda Mathematica. A synopsis of facts, formulae, and methods in elementary mathematics. By W. P. WORKMAN. Clarendon Press, Oxford, 1912. iv + 272 + 28 pages. $1.75.

Although this book was written especially for the use of the English student, any teacher of college mathematics will find it a valuable addition to his library. The author says: "Most students and teachers of mathematics have at times wished for a book to which they might speedily refer for a particular formula or method. Such a want the author attempts to supply in the present volume, which collects in convenient compass the facts, formulae, and results of elementary mathematics and is intended, not to replace existing text-books, but to be a companion to them all, useful for revision and handy for reference."

The words "elementary mathematics" must be interpreted to mean all the subjects usually given in the first three years of an American college course, except integral calculus, which is entirely omitted. Many of the subjects are outlined much more fully than is usual in American texts. Not only are the formulas stated, but in many cases the methods of obtaining them are outlined with sufficient fullness to enable any teacher to complete them. As an illustration of the book's completeness, ten methods of solving numerical equations are given, some of which are entirely unknown to many teachers of college mathematics in this country.

Algebra, including the theory of equations and determinants, and plane and spherical trigonometry occupy the first 100 pages. Another 100 pages are devoted to the subject of geometry, from which the usual high school course in plane and solid geometry is omitted. Besides the ordinary course in analytic geometry, it includes the outline of what is usually given under the name of modern geometry: contact of conics, geometric and analytic treatment of polar reciprocation, harmonic ranges and pencils, inversion, etc.

The portion of the book devoted to the geometric treatment of conics is so complete that it might almost be used as a text-book in that subject, and the application of the differential calculus to the study of curves contains much that is not usually found in American books. The book closes with short outlines of statics, dynamics, and hydrostatics. Although the treatment throughout is distinctly of the English type, any American student specializing in mathematics will find it very useful.

C. H. ASHTON.

ISIS, Revue consacrée a l'histoire de la science, publiée par GEORGE SARTON, D.Sc., Wondelgem-lez-Gand, Belgium.

This proposed journal, to be devoted to the history of science, has secured the support of a notable list of scholars representing almost every phase of academic activity. The "Comité de patronage" includes, as representatives of science, Svante Arrhenius, Jacques Loeb, Wilhelm Ostwald, Sir William Ramsay, Jean Mascart and others; among the representatives of mathematics and the history of mathematics are Moritz Cantor, Siegmund Günther, Sir Thomas L. Heath, H. G. Zeuthen and David Eugene Smith; among orientalists and classical scholars are Franz Cumont, Ludwig Stein and J. L. Heiberg; and among historians, Henri Berr and Karl Lamprecht.

The aim of the journal is to study the genesis and development of scientific ideas as an integral part of the history of civilization. Further than this the journal will assist in assembling the material necessary for such study and in perfecting the methods and instruments of historical research in science. It should appeal not only to historians of science but to all students of history, science, philosophy and sociology; in a word, to all who desire to understand any phase of the evolution of the human intellect.

Contributions are to be published in French, English, German or Italian.

This corresponds to the practise of the *Bibliotheca Mathematica* which is devoted to the history of the mathematical sciences. The editor of *Isis* is George Sarton, Wondelgem-lez-Gand, Belgium. Subscriptions, 30 francs per year for four numbers of about 160 pages each, should be sent to the printer, Soc. an. M. Weissenbruch, imprimeur du Roi, 49, rue du Poinçon, Brussels. The attention of our readers is called to the desirability of supporting such worthy projects for the advancement of science as *Isis* and the *Bibliotheca Mathematica* which latter is published by Teubner, Leipzig, at 20 marks per year.

L. C. KARPINSKI.

An Introduction to Mathematical Physics. By R. A. HOUSTON. Longmans, Green, and Co., London, 1912. x+200 pages. $2.00.

This book is intended as a first survey of the field of mathematical physics, for students already familiar with elementary physics and dynamics. As a single volume of this grade on the whole subject it is something of a novelty, at least in English. To compare with it the reviewer calls to mind only the somewhat similar works by Christiansen and Weinstein. Its scope is, however, more restricted than the title might seem to imply.

There is no consideration given to the molecular theory of matter, the modern electron theory, or the principle of relativity. Historical perspective is absent and the physical setting of the mathematical notions is treated in the briefest way. The book may perhaps be best described as a somewhat detailed syllabus of the classic mathematical physics of the eighteenth and nineteenth centuries, so far as that relates to the descriptive theory of physical media, expressed in differential equations and the related integral formulas.

There are six chapters, entitled: attraction, hydrodynamics, Fourier's series and conduction of heat, wave motion, electromagnetic theory, and thermodynamics. In view of the moderate space employed, the selection of material seems judicious and representative. The mathematical treatment is rather better than the explanations of experimental results. A few of the equations are written in vector form, as an alternative short-hand, but there is confusion in the notation between a vector and its absolute value, and no real use of vector analysis as an instrument of thought. Each chapter contains several lists of examples as exercises, on the whole free from the common fault of possessing merely academic interest; but no hint of a bibliography appears.

A. C. LUNN.

PROBLEMS AND QUESTIONS.

B. F. FINKEL, CHAIRMAN OF THE COMMITTEE.

PROBLEMS FOR SOLUTION.

ALGEBRA.

387. Proposed by H. C. FEEMSTER, York, Neb.

Sum the following series:

$$(1)\ \sum_{n=1}^{n=\infty} \frac{1}{8^{n+1}n!}; \qquad (2)\ \sum_{n=1}^{n=\infty} \frac{1.3.5.\cdots 2n+1}{2^{2n+1}(2n-1)!}; \qquad (3)\ \sum_{n=1}^{n=\infty} \frac{1}{2^{2n}n!}$$

GEOMETRY.

415. Proposed by WILLIAM D. CAIRNS, Oberlin College.

For three points on a straight line, P_1, P_2, P_3, we have

$$P_1P_2 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}; \qquad P_2P_3 = \sqrt{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2};$$

and

$$P_1P_3 = \sqrt{(x_3-x_1)^2+(y_3-y_1)^2+(z_3-z_1)^2},$$

while

$$P_1P_2 + P_2P_3 = P_1P_3.$$

Prove directly that the sum of the first two radical expressions equals the third.

416. Proposed by W. H. BUSSEY, University of Minnesota.

Let AB be a diameter of any circle, and let C be the third vertex of the equilateral triangle of which AB is one side. Divide AB into six equal parts and let P denote the second point of division from A (*i. e.*, $AP = \frac{1}{3}AB$). Draw the straight line CP and produce it to meet the circle in R Prove that AR is one-sixth of the circumference.*

CALCULUS.

337. Proposed by R. P. BAKER, University of Iowa.

Show that for a, b relatively prime integers ,

$$\int_0^1 (\cos 2\pi ax + \cos 2\pi bx)dx = \frac{2}{\pi ab}\left[\frac{a+b}{\sin\frac{\pi}{a+b}} - \frac{a-b}{\sin\frac{\pi}{a-b}}\right],$$

or

$$\frac{1}{\pi ab}\left[(a+b)\cot\frac{\pi}{2(a+b)} - (a-b)\cot\frac{\pi}{2(a-b)}\right]$$

according as a and b are both odd or one of them is even.

---

* *Note of explanation:* This problem was called to my attention by a member of the class in design in our art department. A generalization of the construction can be used to divide a circle approximately into any number of equal parts. If the diameter AB is divided into n equal parts, and if P be the second point of division from A, the line CP produced will meet the circle in R so that the arc AR is an arc of $360°/n$. The construction is theoretically exact when $n = 6$ and in the trivial cases when $n = 2$ or 4. It is used, I am told, when $n = 5$ and 7 and the construction is very approximately correct. The problem for $n = 6$ is not difficult if one uses analytic geometry or trigonometry. Perhaps some reader of the MONTHLY can work it by means of elementary geometry. I asked the student who called the problem to my attention why she did not use the ordinary geometric constructions for dividing a circle into 5 or 6 equal parts. She said she wanted *one* construction that would be good in all cases.

W. H. B.

338. Proposed by RICHARD LOCHNER, Philadelphia, Pa.

An elliptical field has a major axis of 100 feet and a minor axis of 10 feet. A cow is tethered at the end of the major axis and another at the end of the minor axis. If each cow can graze over half the field, how long is the rope of each? What is the area of the portion over which the cows can graze in common?

MECHANICS.

274. Proposed by G. B. M. ZERR.

A sphere moves on the concave side of a rough cylindrical surface of which the transverse section perpendicular to the generating lines is a hypocycloid. If $s = l \sin n\theta$ be the intrinsic equation of the hypocycloid, then $l = (a - b)4b/a$, $n = a/(a - 2b)$, where a = radius of fixed circle, b = radius of rolling circle.

275. W. J. GREENSTREET, Editor Mathematical Gazette, England.

If a particle be attracted towards the angular points of a regular hexagon by forces equal to r^{-5}, at distance r, find the condition for stability of equilibrium.

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

190. Proposed by H. C. FEEMSTER, York, Neb.

Show that, if n is a prime number and r is an integer less than n, then

$$\underline{|r-1}\ \underline{|n-r} + (-1)^{r-1} = M \cdot n,$$

where M is an integer.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

376. Proposed by W. W. BEMAN, Ann Arbor, Mich.

If

$$\left(1+\frac{1}{m}\right)^m/e = 1 - a_1\frac{1}{m} + a_2\frac{1}{m^2} - a_3\frac{1}{m^3} + \cdots,$$

prove that

$$na_n = \sum_{k=1}^{k=n} \frac{k}{k+1}\, a_{n-k}$$

and compute $a_1, a_2, a_3, \cdots, a_8$.

SOLUTION BY THE PROPOSER.

For convenience, put $1/m = x$, and $y = (1 + x)^{1/x}/e$. Then

$$ey = (1 + x)^{1/x},$$

$$\log y = -\frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

and

$$\frac{y'}{y} = -\frac{1}{2} + \frac{2x}{3} - \frac{3x^2}{4} - \cdots.$$

Assume

$$y = 1 - a_1x + a_2x^2 - a_3x^3 + \cdots.$$

Then

$$y' = -a_1 + 2a_2x - 3a_3x^2 + 4a_4x^3 - \cdots$$

$$= \left(1 - a_1x + a_2x^2 - a_3x^3 + \cdots\right)\left(-\frac{1}{2} + \frac{2x}{3} - \frac{3x^2}{4} + \cdots\right).$$

Equating coefficients,

$$a_1 = \tfrac{1}{2}, \qquad 2a_2 = \tfrac{1}{2}a_1 + \tfrac{2}{3},$$
$$3a_3 = \tfrac{1}{2}a_2 + \tfrac{2}{3}a_1 + \tfrac{3}{4},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$na_n = \tfrac{1}{2}a_{n-1} + \tfrac{2}{3}a_{n-2} + \tfrac{3}{4}a_{n-3} + \cdots + \frac{n}{n+1}$$
$$= \sum_{k=1}^{k=n} \frac{k}{k+1} a_{n-k}.$$

The first six values of a_n are

$$a_1 = \tfrac{1}{2}, \quad a_2 = \frac{11}{24}, \quad a_3 = \frac{7}{16}, \quad a_4 = \frac{2447}{5760}, \quad a_5 = \frac{459}{2304}, \quad a_6 = \frac{238043}{580608}.$$

See solution of 377 in the December issue.

Solutions of 377 and 378 were received, too late for credit in the December issue, from C. N. Schmall and H. E. Trefethen; also of 378 from A. L. McCarty.

GEOMETRY.

409. Proposed by S. LEFSCHETZ, University of Nebraska.

The sides of a triangle being in arithmetic progression, a and a' being, respectively, the smallest and largest sides, r, R the radii of the inscribed and circumscribed circles, to prove that $6Rr = aa'$.

SOLUTION BY W. C. EELLS, Tacoma, Wash.

Since the sides are in arithmetic progression, represent them by

$$a, \quad \frac{a+a'}{2}, \quad a'.$$

Then from trigonometry, we have

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad \text{and} \quad R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}$$

where s is the half-sum of the three sides a, b, c.

Substituting $s = \dfrac{3(a+a')}{4}$ and reducing, we have

$$r = \frac{1}{4}\sqrt{\frac{(3a'-a)(3a-a')}{3}}, \quad \text{and} \quad R = \frac{2aa'}{\sqrt{3(3a'-a)(3a-a')}}$$

whence $rR = \dfrac{aa'}{6}$ and $6rR = aa'$.

Also solved by J. Scheffer, C. N. Schmall, Levi S. Shively, A. M. Harding, and A. L. McCarty.

410. Proposed by A. H. HOLMES, Brunswick, Me.

Given a focus and two tangents to an ellipse, prove that the locus of the foot of the normal corresponding to either tangent is a straight line.

No solution has been received.

411. Proposed by C. N. SCHMALL, New York City.

ABCD is a rectangle of known sides. *BC* being produced indefinitely, it is required to draw a straight line from *A* cutting *CD* and *BC* in *X* and *Y*, respectively, so that the intercept *XY* may be equal to a given straight line. (Unsolved in *Educational Times*.)

REMARK BY L. S. SHIVELY, Mt. Morris, Ill.

This problem cannot be solved with straight edge and compasses only, for an arbitrary length of the given straight line. To prove this, it will be sufficient to show that for a particular length the solution, with these restrictions, is impossible.

Let the length of the given line be twice the diagonal of the rectangle. Suppose the problem to have been solved. Let O be the midpoint of XY. Draw AC and CO. Then triangles COY and ACO are isosceles and we have

$$\angle \text{CAO} = \angle \text{AOC} = 2 \angle \text{OYC}$$

and

$$\angle \text{ACB} = \angle \text{CAO} + \angle \text{OYC} = 3 \angle \text{OYC}.$$

Since the given rectangle is arbitrary we have here a solution for the trisection of an arbitrary acute angle, which is known to be impossible with straight edge and compasses. Hence the proposed construction is also impossible.

Solutions were received, too late for credit in the December number, of 403, from W. H. Johnson; of 404, from S. Lefschetz; of 406, 407, and 408 from M. A. Muzzy and H. E. Trefethen.

CALCULUS.

330. Proposed by C. N. SCHMALL, New York City.

X is a homogeneous function, in the nth degree, of x, y, z; Y is any function of u, v, w. If $ux = vy = wz = 1 + X$...(1), prove that (by Eurler's Theorem),

$$x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z} + \frac{n}{x}\frac{\partial Y}{\partial u} + \frac{n}{y}\frac{\partial Y}{\partial v} + \frac{n}{z}\frac{\partial Y}{\partial w} = (n-1)\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right).$$

SOLUTION BY THE PROPOSER.

From (1) we have

$$\log x + \log u = \log y + \log v = \log z + \log w = \log (1 + X)$$

Differentiating, we have

$$\frac{1}{x} + \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{1+X}\frac{\partial X}{\partial x},$$

or

and similarly,

$$\left.\begin{aligned} \frac{x}{u}\frac{\partial u}{\partial x} &= \frac{x}{1+X}\frac{\partial X}{\partial x} - 1 \\ \frac{y}{u}\frac{\partial u}{\partial y} &= \frac{y}{1+X}\frac{\partial X}{\partial y} \\ \frac{z}{u}\frac{\partial u}{\partial y} &= \frac{z}{1+X}\frac{\partial X}{\partial z} \end{aligned}\right\} \tag{2}$$

We get similar expressions involving the partial derivatives of v and w, with respect to x, y and z.
Now

$$x\frac{\partial Y}{\partial x} + y\frac{\partial Y}{\partial y} + z\frac{\partial Y}{\partial z} + \frac{n}{x}\frac{\partial Y}{\partial u} + \frac{n}{y}\frac{\partial Y}{\partial v} + \frac{n}{z}\frac{\partial Y}{\partial w} \equiv \frac{\partial Y}{\partial u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) + \frac{n}{x}\frac{\partial Y}{\partial u} + \frac{n}{y}\frac{\partial Y}{\partial v} + \frac{n}{z}\frac{\partial Y}{\partial w}. \tag{3}$$

From (1),

$$x = \frac{1+X}{u}, \quad y = \frac{1+X}{v}, \quad z = \frac{1+X}{w}. \tag{4}$$

Hence, the right member of (3) can be written, in the form,

$$\frac{\partial Y}{\partial u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) + \frac{nu}{1+X}\frac{\partial Y}{\partial u} + \frac{nv}{1+X}\frac{\partial Y}{\partial v} + \frac{nw}{1+X}\frac{\partial Y}{\partial w}. \tag{5}$$

But by adding the three equations in (2) we have

$$\frac{1}{u}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = \frac{x}{1+X}\frac{\partial X}{\partial x} + \frac{y}{1+X}\frac{\partial X}{\partial y} + \frac{z}{1+X}\frac{\partial X}{\partial z} - 1. \tag{6}$$

Now by the aid of (6) the expression (5) can be written,

$$u\frac{\partial Y}{\partial u}\left(\frac{x}{1+X}\frac{\partial X}{\partial x} + \frac{y}{1+X}\frac{\partial X}{\partial y} + \frac{z}{1+X}\frac{\partial X}{\partial z} - 1\right) + \frac{n}{1+X}\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right) \tag{7}$$

$$= u\frac{\partial Y}{\partial u}\left(\frac{nX}{1+X} - 1\right) + \frac{n}{1+X}\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right) \tag{8}$$

(by applying Euler's Theorem to the first parenthesis).

$$= u\frac{\partial Y}{\partial u}\left(\frac{nX - X - 1}{1+X}\right) + \frac{n}{1+X}\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right),$$

$$= \frac{1}{1+X}\left\{u\frac{\partial Y}{\partial u}(nX - X - 1) + n\left(u\frac{\partial Y}{\partial u} + v\frac{\partial Y}{\partial v} + w\frac{\partial Y}{\partial w}\right)\right\}.$$

Remark.—The right member of the result I first gave is apparently incorrect.

Also solved by Elmer Schuyler.

331. Proposed by T. H. GRONWALL, Chicago, Illinois.

To show that:

(1) $\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right) = \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy,$

(2) $\frac{d^n}{dx^n}\left(\frac{1-\cos x}{x}\right) = \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n}{2}\pi\right) dy.$

SOLUTION BY ELIJAH SWIFT, Princeton, N. J.

Assume that the formula is true for $n-1$.

Then

$$\frac{d^{n-1}}{dx^{n-1}}\left(\frac{\sin x}{x}\right) = \frac{1}{x^n}\int_0^x y^{n-1} \sin\left(y + \frac{n}{2}\pi\right) dy.$$

Differentiating both sides and using the formula we have to prove

$$(A) \quad \frac{1}{x^{n+1}}\int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy = -\frac{n}{x^{n+1}}\int_0^x y^{n-1} \sin\left(y + \frac{n}{2}\pi\right) dy + \frac{1}{x^n}\cdot x^{n-1} \sin\left(x + \frac{n}{2}\pi\right).$$

or

$$(B) \quad \int_0^x y^n \sin\left(y + \frac{n+1}{2}\pi\right) dy = -n\int_0^x y^{n-1} \sin\left(y + \frac{n\pi}{2}\right) dy + x^n \sin\left(x + \frac{n}{2}\pi\right).$$

Differentiating B as to x,

$$(C) \quad x^n \sin\left(x + \frac{n+1}{2}\pi\right) = -nx^{n-1} \sin\left(x + \frac{n\pi}{2}\right) + nx^{n-1} \sin\left(x + \frac{n\pi}{2}\right) + x^n \cos\left(x + \frac{n\pi}{2}\right).$$

which is obviously an identity.

Hence (B) is an identity, except for an additive constant. This constant must be zero, as we see if we let x approach zero. (A) follows from (B). Then since the given formula holds for $n = 0$, it holds for all values of n.

Formula (2) may be derived in the same manner.

Also solved by A. M. Harding, W. C. Eells and the Proposer.

NEWS AND NOTES.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

The March number of *Popular Science Monthly* contains an article on "Henri Poincaré as an Investigator" by PROFESSOR JAMES B. SHAW.

PROFESSOR G. A. MILLER, of the University of Illinois, will teach in the summer session of the University of California.

A campaign is in progress to raise a fund of about $100,000 for the purpose of endowing a second professorship of natural philosophy at the University of Edinburgh, as a memorial to PROFESSOR PETER GUTHRIE TAIT who died in 1901, after a long career as professor at Edinburgh. To mathematicians Tait is best known for his book on quaternions and his topological researches on knots.

PROFESSOR CONSTANTIN CARATHÉODORY has been appointed professor of mathematics at the University of Göttingen as successor to Professor Felix Klein. Carathéodory made his doctorate at Göttingen in 1904, was privatdozent at Göttingen and Bonn, then professor at Hanover and Breslau.

PROFESSOR VIVANTI, University of Pavia, Italy, has recently published a volume of about 500 pages, entitled "Esercizi di analisi infinitesimale." The book is composed of 575 problems and their solutions. The problems are chosen especially with a view to applications in mechanics and in physics, and more than two-thirds of them are said to be new. A large number of these exercises relate to finding the derivative or the integral. The work closes with solutions of differential equations and a few problems in the calculus of variations. The arrangement of the material is similar to that of the "Lezioni di analisi infinitesimale," which was published by the same author in 1911.

The following suggestion from a new subscriber, whose attention was arrested by the Foreword in the January issue, may contain some pedagogical significance: ". . . In short why not get contributions from those who do not like mathematics, telling just what it was about their college course in mathematics which was distasteful to them. This might include not only college students at the present time in advanced classes, but also men and women who have excelled in other lines but did not prefer mathematics. Such contributions, of course, would be of little interest to mathematicians, but they would no doubt bring to light some matters very useful to those concerned with the teaching of mathematics. I am one who believes the popular distaste for mathematics is brought about largely by poor teaching methods, and that in the large majority of cases the students' dislike may be traced to instructors' methods rather than to anything in the mathematics itself." A really scientific investigation along this line might lead to valuable results.

See also the leading article in this issue, entitled "Some things we wish to know."

Attention is called to the advertisements of publishers and others in the various issues of the MONTHLY. The EDITORS desire to express their appreciation of the interest shown by the representatives of these firms in thus contributing toward the support of this journal. We trust that our readers will reciprocate this interest by referring to these announcements whenever possible.

SUMMER COURSES IN MATHEMATICS FOR 1913.

UNIVERSITY OF CHICAGO. June 16 to August 29. See the February issue.

UNIVERSITY OF ILLINOIS. June 16 to August 8.—By Dr. Crathorne, College

algebra (6 hours), Calculus of variations (6 hours); By Dr. Börger, Trigonometry (5 hours), Theory of equations (6 hours); By Mr. Rutledge; Analytic geometry (10 hours); By Mr. Rowland, First course in calculus (10 hours); By Mr. Carscallen, Second course in calculus (6 hours). These courses are equivalent in credit and in number of class room hours to the corresponding courses given in the regular university year.

UNIVERSITY OF KANSAS. June 12 to July 23.—By Assistant Professor Duval, Solid geometry (2 hours), Higher algebra II (3 hours, graduate credit); By Associate Professor Van der Vries, Plane trigonometry (2 hours), Projective geometry (2 hours, graduate credit); By Assistant Professor Jordan, Analytic geometry I (2 hours), Calculus I (3 hours). It is planned to offer a series of six different graduate courses in three consecutive summer sessions, giving a total of 15 hours credit toward the A.M. degree.

UNIVERSITY OF MICHIGAN. June 21 to August 22.—By Professor W. W. Beman, Differential equations, Geometry and algebra (Teachers' Course); By Professor J. L. Markley, Theory of functions, Differential calculus; By Professor W. B. Ford; Infinite series and products, Harmonic analysis; By Professor P. Field; Projective geometry, College algebra; By Professor T. R. Running, Differential and integral calculus; By Professor L. C. Karpinski, History of mathematics (arithmetic and algebra), Analytic geometry (double course); By Mr. Escott, Theory of finance, insurance and statistics, Trigonometry; By Mr. Garretson, Solid geometry, Solid analytic geometry; By Mr. Coe, Elementary algebra, Plane geometry; By Mr. Hopkins, Calculus.

UNIVERSITY OF MINNESOTA.—By Dean J. F. Downey, Higher algebra, part I, Higher algebra, part II; By Professor W. E. Brooke, Trigonometry and analytic geometry, Differential and integral calculus; By Professor W. H. Kirchner, Solid geometry, Descriptive geometry; By Dr. H. L. Slobin, Trigonometry, Analytic geometry.

UNIVERSITY OF WISCONSIN. June 23 to August 1.—By Professor Van Vleck, College algebra, Axioms of geometry, Principal functions of mathematics and physics; By Assistant Professor Hart, Elementary solid geometry, Analytic geometry, The teaching of secondary mathematics; By Assistant Professor Dowling, Plane trigonometry and logarithms, Calculus, Introduction to higher plane curves; By Assistant Professor Dresden, Theory of complex numbers and algebraic equations, Differential equations, Calculus of variations; By Mr. Burgess, Elementary analysis (begins June 16), Elementary theory of transformations; By Mr. Taylor, Integral calculus (begins June 19).

THE UNIVERSITY OF CALIFORNIA.—By Assistant Professor J. H. McDonald, Solid geometry, Theory of functions of a complex variable; By Associate Professor D. N. Lehmer, Plane analytic geometry, Differential calculus; By Professor G. A. Miller, Plane trigonometry, Theory of algebraic equations; By Mr. B. A. Bernstein, Graphic algebra, Integral calculus; By Miss Thirmuthis Brookman, Mathematical teaching in the high school, Teachers' course in mathematics for secondary schools.

5

THE
AMERICAN
MATHEMATICAL MONTHLY

VOLUME XX MAY, 1913 NUMBER 5

THE FOUNDATION PERIOD IN THE HISTORY OF GROUP THEORY.

By JOSEPHINE E. BURNS, University of Illinois.

The Earliest Group Notions.—Henri Poincaré has pointed out that the fundamental conception of a group is evident in Euclid's work; in fact that the foundation of Euclid's demonstrations is the group idea. Poincaré establishes this assertion by showing that such operations as successive superposition and rotation about a fixed axis presuppose the displacements of a group. However much the fundamental group notions were unconsciously used in the work of early mathematicians, it was not until the latter part of the eighteenth century that these notions began to take life and develop.

The Foundation Period.—The period of foundation of group theory as a distinct science extends from Lagrange (1770) to Cauchy (1844–1846), a period of seventy-five years. We find Lagrange considering the number of values a rational function can assume when the variables are permuted in every possible way. With this beginning the development may be traced down through the contributions of Vandermonde, Ruffini, Abbati, Abel, Galois, Bertrand and Hermite, to Cauchy's period of active production (1844–1846). At the beginning of this period group theory was a discovery useful in the theory of equations; at the end it existed as a distinct science, not yet, to be sure, entirely free but so nearly so that this may be called the close of the foundation period.

LAGRANGE, RUFFINI, GALOIS.

Lagrange.—The contributions of Lagrange are included in his memoir, *Reflexions sur la résolution algébrique des equations*, published in the Mémoirès of the Academy of Science at Berlin in 1770–1771.[1] In this paper Lagrange first applies what he calls the "calcul des combinaisons" to the solution of algebraic equations. This is practically the theory of substitutions, and he uses it to show wherein the efforts of his predecessors, Cardan, Ferrari, Descartes, Tschirnhaus, Euler and Bezout fail in the case of equations of degree higher than the

[1] *Œuvres de Lagrange*, vol. 3, pp. 204–420.